TORELLI THEOREM VIA FOURIER-MUKAI TRANSFORM.

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We show that the Fourier transform on the Jacobian of a curve interchanges "δ functions" on the curve and the theta divisor. The Torelli theorem is an immediate consequence.

1. Statement of the theorem.

1.1. We live over an algebraically closed base field k. Let J be an abelian variety equipped with a principal polarization $\theta : J \xrightarrow{\sim} J^{\circ} = \mathrm{Pic}^0(J)$, so we have the corresponding Fourier transform $\mathcal{F}$ on the derived category of quasi-coherent sheaves $D(J, \mathcal{O})$ (see [6]).

Let Θ be the theta divisor in J. Notice that Θ is defined up to translation, and any non-trivial translation does not preserve Θ. So we may consider Θ as a canonically defined algebraic variety equipped with a J-torsor of embeddings $j : \Theta \hookrightarrow J$; we call these j's *standard embeddings*. Denote by Θ^{ns} the open subset of smooth points of Θ. For a standard embedding j let $j^{ns} : \Theta^{ns} \hookrightarrow J$ be its restriction to Θ^{ns}.

Our Θ carries a canonical involution $x \mapsto x^{\nu}$; this is the unique involution such that for any standard embedding j the embedding $j^{\nu} : x \mapsto -j(x^{\nu})$ is also standard. For a line bundle L on Θ or Θ^{ns} set $L^{\nu} := \nu^* L$.

The pull-back j^*F of an $\mathcal{O}_J$-module F does not change if we translate both j and F by the same element of J. Thus the image of $j^{ns*} : \mathrm{Pic}(J) \to \mathrm{Pic}(\Theta^{ns})$ is a canonically defined subgroup of $\mathrm{Pic}(\Theta^{ns})$ (it does not depend on j). Denote by $A(J)$ the corresponding quotient group.

Let $\mathcal{T} \subset \mathrm{Pic}(\Theta^{ns})$ be the subset of line bundles L such that

(i) $L \cdot L^{\nu} = \omega_{\Theta^{ns}}$

(ii) $A(J)$ is generated by the image of L.

Remark. Since the tangent bundle to J is trivial, one has $\omega_{\Theta^{ns}} = j^{ns*}\mathcal{O}_J(j(\Theta))$. Thus, if $\mathcal{T}$ is non-empty then ν acts on $A(J)$ as -1.

1.2. From now on we assume that J is the Jacobian of a smooth projective curve C of genus $g \geq 2$ equipped with the canonical polarization. There is a standard embedding $i : C \hookrightarrow J$ defined up to translation; the standard embeddings i form a J-torsor isomorphic to $\mathrm{Pic}^{-1}(C)$.

1.3. **Theorem.** The set $\mathcal{T}$ is non-empty. For any $L \in \mathcal{T}$ and a standard embedding $j : \Theta \hookrightarrow J$ the Fourier transform $\mathcal{F}(j_*^{ns} L)$ equals to $(\pm i)_*(M)[1-g]$ where $i : C \hookrightarrow J$ is a standard embedding, M is a line bundle of degree $g-1$ on C.

2. Proof of the theorem.

2.1. Let us start with some preliminaries.

Consider J as the moduli space[1] of line bundles of degree 0 on C. A line bundle E of degree $1-g$ yields a standard embedding $j = j_E : \Theta \hookrightarrow J$. Namely, there is a canonical morphism $\sigma : \mathrm{Sym}^{g-1} C \to \Theta$ such that $j_E \sigma$ sends a divisor D to $E(D)$. Thus the J-torsor of j's equals $\mathrm{Pic}^{1-g} C$. The above σ is an isomorphism over Θ^{ns}; we denote by α the inverse open embedding $\Theta^{ns} \hookrightarrow \mathrm{Sym}^{g-1} C$.

For a group T equipped with an involution ν we denote by $\widetilde{T}$ the corresponding semi-direct product of T and $\mathbb{Z}/2\mathbb{Z}$. Our $\mathcal{T}$ carries a canonical action of $\widetilde{J}$ (here ν acts on J as -1). Namely, an element $l \in J$ acts as tensor product by the line bundle $j^{ns*}\theta(l)$ (notice that, since $\theta(l)$ is translation invariant, this line bundle does not depend on j and is ν-anti-invariant), and $\mathbb{Z}/2\mathbb{Z}$ acts by ν.

2.2. **Proposition.** The $\widetilde{J}$-action on $\mathcal{T}$ is transitive.

We prove 2.2 in 2.7. Notice that for $g = 2$ we have $\Theta = \Theta^{ns} = C$, so here the proposition is clear.

The map $(L, j) \mapsto j_*^{ns} L$ commutes with the action of $\widetilde{J}$; here $\widetilde{J}$ acts on $\mathcal{O}$-modules on J by twists by degree 0 line bundles and -1 symmetry. The Fourier transform interchanges twists by degree 0 line bundles and translations and commutes with the -1 symmetry. Since the set of embeddings $\pm i : C \hookrightarrow J$ is a $\widetilde{J}$-torsor, 2.2 implies that it suffices to prove our theorem for a single pair (L, j).

Take any pair (i, M) where $i : C \hookrightarrow J$ is a standard embedding, $M \in \mathrm{Pic}^{g-1} C$. The theorem follows immediately from the involutivity property of $\mathcal{F}$ and the following fact:

[1] We consider J as a plain variety ignoring the stack structure

2.3. **Proposition.** One has $\mathcal{F}(i_* M)[1] = j^{ns}_* L$ where $j := j_{M^{-1}}$ and $L \in \mathcal{T}$.

We prove 2.3 in 2.8.

For every point $x \in C$ and every $d \geq 2$ consider the embedding

$$a^d_x : \operatorname{Sym}^{d-1} C \to \operatorname{Sym}^d C : D \mapsto D + x.$$

Let us denote by R^d_x the image of a^d_x.

2.4. **Lemma.** For $d \geq 2$ there is an exact sequence of abelian groups

$$0 \longrightarrow \operatorname{Pic}(J) \xrightarrow{(j\sigma)^*} \operatorname{Pic}(\operatorname{Sym}^d C) \xrightarrow{\deg} \mathbb{Z} \longrightarrow 0$$

where the homomorphism deg is normalized by the condition that $\deg(\mathcal{O}(R^d_x)) = 1$ for any $x \in C$. If $\mathbb{P} \subset \operatorname{Sym}^d C$ is a complete linear system of positive dimension then $\deg(L) = \deg(L|_{\mathbb{P}})$.

Proof of the lemma[2]. For d sufficiently large the statement is true since $\operatorname{Sym}^d C$ is a projective bundle over J. Let $\pi_d : C^d \to \operatorname{Sym}^d C$ be the canonical projection. Then we have $\pi^*_d \mathcal{O}(R^d_x) \simeq \mathcal{O}_C(x) \boxtimes \ldots \boxtimes \mathcal{O}_C(x)$. Therefore, $\mathcal{O}(R^d_x)$ is ample and $H^i(\operatorname{Sym}^d C, \mathcal{O}(-nR^d_x)) = 0$ for $n > 0$, provided that $i \leq 1$, $d \geq 2$ or $i \leq 2$, $d \geq 3$ (this follows from the symmetric Kunneth formula proved in [7], exp. XVII, 5.5.34). It follows from the Lefschetz theorem for Picard groups (see [5], exp.11 (3.12), exp.12 (3.4)) that the map $a^{d*}_x : \operatorname{Pic}(\operatorname{Sym}^d C) \to \operatorname{Pic}(\operatorname{Sym}^{d-1} C)$ is an isomorphism for $d > 3$ and is an embedding for $d = 3$. It remains to check that a^{3*}_x is surjective. Let us denote by $K \subset \operatorname{Pic}(C \times C)$ the subgroup of line bundles L such that the restrictions $L|_{x \times C}$ and $L|_{C \times x}$ are trivial. Then there is an isomorphism $u : \operatorname{End}(J) \xrightarrow{\sim} K : \phi \mapsto (i_x \times \phi i_x)^* \mathcal{P}$ where $i_x : C \to J$ is the embedding corresponding to x, $\mathcal{P}$ is the Poincaré line bundle on $J \times J$. Let $\operatorname{Pic}^+(C \times C)$ be the subgroup of line bundles stable under the involution $(x_1, x_2) \mapsto (x_2, x_1)$, let $K^+ = K \cap \operatorname{Pic}^+(C \times C)$. Then u induces an isomorphism of $\operatorname{End}^+(J)$ with K^+ where $\phi \in \operatorname{End}^+(J)$ if and only if ϕ is self-dual. Let $r : \operatorname{Pic}(C \times C) \to K$ be the homomorphism given by $r(F) = F \otimes [(F^{-1}|_{C \times x}) \boxtimes (F^{-1}|_{x \times C})]$. Then $r(\operatorname{Pic}^+(C \times C)) = K^+$ and we have the following commutative diagram

$$(1) \qquad \begin{array}{ccccc} \operatorname{Pic}(J) & \xrightarrow{\sigma_2^*} & \operatorname{Pic}(\operatorname{Sym}^2 C) & \xrightarrow{\pi_2^*} & \operatorname{Pic}^+(C \times C) \\ \downarrow s & & & & \downarrow r \\ \operatorname{End}^+(J) & & \xrightarrow{u} & & K^+ \end{array}$$

[2]An alternative proof can be found in [4]

where $\sigma : \mathrm{Sym}^2 C \to J$ maps D to $\mathcal{O}(D-2x)$, $s : \mathrm{Pic}(J) \to \mathrm{End}^+(J)$ is the standard homomorphism $L \mapsto \phi_L$ where $\phi_L(a) = t_a^* L \otimes L^{-1}$. Since s is surjective it follows that the composition $r \circ \pi_2^* \circ \sigma_2^*$ is surjective. Thus, in proving that some line bundle $L \in \mathrm{Pic}(\mathrm{Sym}^2 C)$ comes from $\mathrm{Pic}(\mathrm{Sym}^3 C)$ we may assume that $\pi_2^* L$ belongs to the kernel of r. In other words, $\pi_2^* L \simeq L_1 \boxtimes L_1$ for some line bundle L_1 on C. The S_2-action on $L_1 \boxtimes L_1$ either coincides with the standard one or differs from it by -1. In accordance with this dichotomy we equip $\widetilde{L} = L_1 \boxtimes L_1 \boxtimes L_1$ either with the standard S_3-action or with the standard action twisted by the sign character. Then if we consider $\widetilde{L}$ as a line bundle on $\mathrm{Sym}^3 C$ we have $a_x^{3*}\widetilde{L} = L$.

2.5. **Lemma.** Assume that C is hyperelliptic. Let $Q \subset \mathrm{Sym}^{g-1} C$ be the complement to the image of α. Then Q is an irreducible divisor and $\deg(Q) = -2$.

Proof of the lemma.

Let $\tau : C \to C$ be the hyperelliptic involution. For every $d > 1$ let us denote by $Q_d \subset \mathrm{Sym}^d C$ the reduced effective divisor consisting of D such that D contains a divisor of the form $x + \tau x$. Note that $Q_2 \simeq \mathbb{P}^1$ while Q_d is just the image of $Q_2 \times \mathrm{Sym}^{d-2} C$ under the natural map $\mathrm{Sym}^2 C \times \mathrm{Sym}^{d-2} C \to \mathrm{Sym}^d C$, so it is irreducible. We have $Q = Q_{g-1}$. It is easy to check that

$$a_x^{d*}\mathcal{O}(Q_d) \simeq \mathcal{O}(Q_{d-1} + R_{\tau(x)}^{d-1}). \tag{2}$$

On the other hand, for any points $x, y \in C$ one has

$$a_x^{d*}\mathcal{O}(R_y^d) \simeq \mathcal{O}(R_y^{d-1}). \tag{3}$$

Consider the embedding $a : \mathbb{P}^1 \simeq Q_2 \hookrightarrow \mathrm{Sym}^d C$ given by $D \mapsto D + D_0$ where $D_0 = x_1 + \ldots + x_{d-2}$ is a fixed effective divisor of degree $d-2$. Then by induction we derive from (2) and (3) that $a^*\mathcal{O}(R_y^d) \simeq \mathcal{O}_{\mathbb{P}^1}(1)$ while $a^*\mathcal{O}(Q_d) \simeq \mathcal{O}_{\mathbb{P}^1}(Q_2 \cdot Q_2 + d - 2)$, where $Q_2 \cdot Q_2$ is the self-intersection index of Q_2 in $\mathrm{Sym}^2 C$. Since $Q_2 \cdot Q_2 = 1 - g$ we obtain $a^*\mathcal{O}(Q_d) \simeq \mathcal{O}_{\mathbb{P}^1}(d - g - 1)$. In particular, $a^*\mathcal{O}(Q_{g-1}) \simeq \mathcal{O}_{\mathbb{P}^1}(-2)$ as required.

2.6. **Corollary.** Assume that $g \geq 3$. Then the map $j^{ns*} : \mathrm{Pic}(J) \to \mathrm{Pic}(\Theta^{ns})$ is injective. The group $A(J) = \mathrm{Pic}(\Theta^{ns})/j^{ns*}(\mathrm{Pic}(J))$ is isomorphic to $\mathbb{Z}$ if C is non-hyperelliptic, and to $\mathbb{Z}/2\mathbb{Z}$ otherwise.

Proof of the corollary. (i) Assume that C is non-hyperelliptic. Then by Martens theorem the complement to the image of α has codimension

> 1 (see [2], IV 5.1)[3], so $\alpha^* : \mathrm{Pic}(\mathrm{Sym}^{g-1}C) \xrightarrow{\sim} \mathrm{Pic}(\Theta^{ns})$. We are done by Lemma 2.4.

(ii) Assume that C is hyperelliptic. Then Lemma 2.5 implies that $\mathrm{Pic}(\Theta^{ns})$ is isomorphic to $\mathrm{Pic}(\mathrm{Sym}^{g-1} C)/\mathbb{Z}[Q]$ where $\deg([Q]) = 2$, so our statement follows easily from Lemma 2.4.

2.7. *Proof of Proposition 2.2.*

Choose j to be symmetric, $j^\nu = j$; then $j^{ns*} : \mathrm{Pic}J \to \mathrm{Pic}(\Theta^{ns})$ commutes with the involution. Take $L, L' \in \mathcal{T}$. Since L generates $A(J)$ and $L^\nu \equiv L^{-1} \mod j^{ns*}\,\mathrm{Pic}(J)$ we have either $L \equiv L' \mod j^{ns*}\,\mathrm{Pic}(J)$ or $L^\nu \equiv L' \mod j^{ns*}\,\mathrm{Pic}(J)$. Replacing L by L^ν if necessary we obtain that $L^{-1} \cdot L' \simeq j^{ns*}\xi$ for some $\xi \in \mathrm{Pic}(J)$. Since $\xi^\nu = \xi^{-1}$ we deduce that $\xi \in \mathrm{Pic}^0(J) = J$ which implies the proposition.

Remark. We actually proved that if C is non-hyperelliptic then $\mathcal{T}$ is a $\widetilde{J}$-torsor, while for hyperelliptic C, it is a J-torsor (we did not prove that it is non-empty as yet).

2.8. *Proof of Proposition 2.3.* Our $F := \mathcal{F}(i_*M)[1]$ vanishes outside of $j(\Theta)$ where $j = j_{M^{-1}}$. Since F is the push-forward of a (shifted) line bundle on $C \times J$ one may represent it as the cone of a morphism $f : V_1 \to V_0$ of vector bundles on J. Therefore f is injective (so $F = \mathrm{Coker}\, f$), and for any closed subset $Y \subset J$ of codimension > 2 one has $\mathcal{H}^1_Y F = 0$. Note also that $j(\Theta)$ is precisely the zero locus of $\det(f)$ (this follows from the well-known determinantal description of Θ, see e.g. [2]). On the other hand, it is easy to see that $\det(f)$ annihilates $\mathrm{Coker}\, f$. Therefore, $F = j_*j^*F$. Since the codimension in J of the singular locus of Θ is > 2 this shows that $F = j^{ns}_* L$ where $L := j^{ns*}F$. By Riemann's theorem on singularities of Θ the fibers of L at all closed points of Θ^{ns} are one-dimensional. Since Θ is reduced (see [2] IV.4.5) we see that L is a line bundle on Θ^{ns}.

It remains to prove that $L \in \mathcal{T}$. It is clear that the derived pull-back $\Phi := Lj^{ns*}F$ is a complex with $H^0\Phi = L$, $H^{-1}\Phi = L \otimes \mathcal{O}_J(-j(\Theta))$, other cohomology are 0. On the other hand, computing Φ using base change and the definition of F we see that for $x \in \Theta^{ns}$ the corresponding fibers are $H^0\Phi_{j(x)} = H^1(C, E_x \otimes M)$, $H^{-1}\Phi_{j(x)} = H^0(C, E_x \otimes M)$ where $E_x := i^*\theta(j(x))$ is the line bundle on C of degree 0 that corresponds to $j(x)$. Since $E_{x^\nu} \simeq E_x^{-1} \otimes \omega_C \otimes M^{-2}$, the Serre duality yields $(H^0\Phi_{j(x^\nu)})^* = H^{-1}\Phi_{j(x)}$. Therefore, $L^{\nu*} = L \otimes \mathcal{O}_J(-j(\Theta))$. We see that condition (i) from 1.1 is satisfied.

Let us check condition (ii). We may assume that i corresponds to a line bundle $\mathcal{O}_C(-x)$, $x \in C$. Consider the universal divisor

[3] The proof given in [2] works in arbitrary characteristic.

$\mathcal{D} \subset C \times \mathrm{Sym}^{g-1} C$. Then the pull-back of the Poincaré line bundle on $J \times J$ by the morphism $(i \times (j\sigma)) : C \times \mathrm{Sym}^{g-1} C \to J \times J$ is isomorphic to $p_1^* M^{-1}(\mathcal{D} - C \times R_x)$. It follows that the line bundle L^{-1} on Θ^{ns} is $\alpha^* p_{2*}(\mathcal{O}(\mathcal{D}))(-R_x)$ where $p_2 : C \times \mathrm{Sym}^{g-1} C \to \mathrm{Sym}^{g-1} C$ is the projection. The canonical morphism $\mathcal{O}_{\mathrm{Sym}^{g-1} C} \to p_{2*}(\mathcal{O}(\mathcal{D}))$ is an isomorphism over $\alpha(\Theta^{ns})$. Therefore, $L^{-1} = \alpha^* \mathcal{O}(-R_x)$ which generates $A(J)$, so we are done.

3. Concluding remarks.

3.1. From the Lefschetz theorem for Picard groups (see [5], exp.11 (3.12), exp.12 (3.4)) one can easily derive that the restriction map $j^* : \mathrm{Pic}(J) \to \mathrm{Pic}(\Theta)$ is an isomorphism for an arbitrary principally polarized abelian variety J of dimension $g \geq 4$. Furthermore, if the dimension of the singular locus $\mathrm{Sing}\,\Theta$ is $< g-4$ then Θ is locally factorial as follows from [5], exp.11 (3.14). Hence, in this case $\mathrm{Pic}(\Theta) = \mathrm{Pic}(\Theta^{ns})$ (since the notions of Cartier divisors and Weil divisors on Θ coincide) and $A(J) = 0$. Notice that if J is a Jacobian then the dimension of $\mathrm{Sing}\,\Theta$ is $\geq g-4$. Moreover, Andreotti and Mayer proved in [1] that the closure of the locus of Jacobians constitute an irreducible component of the locus N_{g-4} of principally polarized abelian varieties with $\dim \mathrm{Sing}\,\Theta \geq g-4$. In [3] Beauville established that in the case $g = 4$ the locus N_0 has two irreducible components. He also proved (assuming that the characteristic is zero) that a generic point of N_0 which is not contained in the closure of the locus of Jacobians corresponds to an abelian variety J with $\mathrm{Sing}\,\Theta$ consisting of one ordinary double point (see [3], 7.5). It follows that the corresponding group $A(J)$ is either zero or isomorphic to $\mathbb{Z}$ and the involution acts on $A(J)$ as identity. Therefore, in this case either $A(J) = 0$ or the set $\mathcal{T}$ is empty. The natural question is whether in higher dimensions one still has either $A(J) = 0$ or $\mathcal{T} = \emptyset$ for principally polarized abelian varieties which are not in the closure of the locus of Jacobians.

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